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Thermodynamics of scalar fields in Kerr's geometry

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Abstract: *The one-loop contributions to the entropy for a massive scalar field in a Kerr black hole are investigated using an approximation of the metric, which, after a conformal transformation, permits to work in a Rindler-like spacetime. Of course, as for the Schwarzschild case, the entropy is divergent in the proximity of the event horizon.*

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1 Introduction

As has been recently stressed in a series of papers (see for example [1] and references cited therein), the Bekenstein-Hawking entropy [2–4], that for a stationary black hole can be computed by several methods, which lead to the celebrated result $A/4$ (tree-level contribution) (see for example [5]), has a thermodynamical origin in the sense that it can be defined by the response of the free energy of the black hole to the change of the equilibrium temperature. Such a temperature depends on the parameters of the black hole and may be determined by requiring the smoothness of the related Euclidean solution [4] (on-shell computation).

The situation is completely different if one tries to investigate the entropy within a statistical-mechanical approach, that is by counting the quantum states of the black hole. In fact in this case one is forced to work at an arbitrary temperature, which is not the equilibrium one (off-shell computation). The first computation of this kind has been appeared in the 't Hooft seminal paper [6], where the black hole degrees of freedom have been identified with the ones of a quantum gas of scalar particles propagating outside the horizon at a given temperature T. After that, the 't Hooft brick wall model has been extended to other geometries [7] and to other fields [8] and also a lot of different methods have been proposed in order to compute the entropy of fields in the black hole geometry (see for example Refs. [1, 9, 10] and references cited therein).

As is well known, independently of the method of computation used, the statistical-mechanical quantities were found to be divergent. These divergences are not totally unexpected. In fact their physical origin may be derived by the equivalence principle, which implies that a system in thermal equilibrium has a local Tolman temperature given by $T(x) = T_\infty/\sqrt{|g_{00}(x)|}$, T_∞ being the temperature measured by the observer at the spatial infinity. The leading term in the high temperature expansion for the free energy of a massless quantum gas in a 4-dimensional static space-time is proportional to the integral of $T^4(x)$ over the space variable and this is divergent since the metric has non integrable singularities on the horizon. The nature of these divergences depends on the poles and zeros of the metric. For example, for extreme black holes, where g_{00}

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has higher order zeros, the divergences are much more severe than those which appear in the non extremal case (see for example [11]).

In the present paper we shall compute the one-loop contribution to the entropy of a Kerr black hole due to a massive scalar field. Such a problem has been already studied in Ref. [7] using a semiclassical approach and, for the massless case, in Ref. [12] using the method of "blunt conical singularity". Here we shall compute the free energy through the Euclidean path integral and using heat kernel and ζ-function regularisation methods. In order to perform explicit calculations, we have to use an approximation for the Kerr metric, valid in the proximity of the horizon, which, after a conformal transformation, permits to work in a Rindler-like manifold.

At first sight, the present approach could be confused with the one proposed in Ref. [12]. In fact, in that paper the authors use path integral and heat kernel techniques too and also an approximation for the metric very similar to our, but the use of it and its interpretation is different. Moreover in that paper, the singularity, which naturally appears when arbitrary temperature is considered, is regularised by means of a family of smooth manifolds, while here we prefer to work in the original manifold with the conical singularity, since heat kernel and ζ-function are well known in such kind of spaces.

In the paper we shall use conformal transformation techniques, which are resumed in Sec. 2 and ζ-function regularisation on Rindler-like manifold, which we recall in Sec. 3. In Sec. 4 we propose an approximation of the Kerr metric, valid in the proximity of the horizon and finally, in Sec. 5, we compute the entropy for a massive scalar field in the Kerr geometry and conclude with some comments in Sec. 6.

2 Conformal transformations

Conformal transformation techniques have been used by many authors in order to transform the original static manifold in an ultrastatic one (optical manifold) [13–19]. In the context of black holes, they have been used in Refs. [9, 20, 21] and in Rindler space-times in Refs. [22–24].

In the following we do not work in the optical manifold, but nevertheless we shall use conformal transformations in order to simplify the metric. A conformal transformation does not modify the temperature dependent part of the free energy of the system and so one can compute all thermodynamical quantities in the transformed metric and then simply write them in the original one.

To start with, we consider a scalar field on a 4-dimensional static space-time with metric $g_{\mu\nu}(\mathbf{x})$ and signature $\{-+++\}$ $(\mu,\nu=0,...,3)$. The one-loop partition function at temperature $T=1/\beta$ is given by (as usual we perform the Wick rotation $x_0=-i\tau$ and assume the field to be periodic in the τ variable, with period β)

$$Z_\beta=\int d[\phi]\,\exp\left(-\frac{1}{2}\int \phi L_4\phi\,d^4x\right)\,, \tag{2.1}$$

where ϕ is a scalar density of weight $-1/2$ and L_4 is a Laplace-like operator on the 4-dimensional manifold. It has the form

$$L_4=-\Delta_4+m^2+\xi R\,. \tag{2.2}$$

Here Δ_4 is the Laplace-Beltrami operator in the metric g, m (the mass) and ξ arbitrary parameters and R the scalar curvature of the manifold.

Now we perform the conformal transformation

$$\bar g_{\mu\nu}(\mathbf{x})=e^{2\sigma(\mathbf{x})}g_{\mu\nu}(\mathbf{x})\,, \tag{2.3}$$

$$L_\sigma=\bar L_4=e^{-\sigma}L_4e^{-\sigma}=-\bar\Delta_4+\frac{1}{6}\bar R+e^{-2\sigma}\left[m^2+\left(\xi-\tfrac{1}{6}\right)R\right]\,, \tag{2.4}$$

where $\sigma(\mathbf{x})$ is a suitable function (we shall use over-bar symbols for quantities related to the metric $\bar{g}$). The one-loop partition function transforms as

$$\bar{Z}_\beta = J[g,\bar{g}]\, Z_\beta \,, \tag{2.5}$$

where $J[g,\bar{g}]$ is the Jacobian of the conformal transformation. Such a Jacobian can be explicitly computed [19], but for our purposes it is sufficient to know that in a static manifold it depends linearly on β. In fact it can be expressed as an integral over spacetime and over s of a Seeley-DeWitt coefficient related to the field operator $L_{s\sigma}$ [17, 19, 25]. Since the manifold is static, the integral over the imaginary time trivially gives a β-factor. This means that the Jacobian can be ignored in the computation of thermodynamical quantities starting from the free energy, which is related to the canonical partition function by means of the usual relation

$$F_\beta = -\frac{1}{\beta}\ln Z_\beta = -\frac{1}{\beta}\left(\ln \bar{Z}_\beta - \ln J[g,\bar{g}]\right) \,, \qquad \bar{F}_\beta = -\frac{1}{\beta}\ln \bar{Z}_\beta \,. \tag{2.6}$$

From the latter equation one obtains

$$S_\beta = \beta^2 \partial_\beta F_\beta = \beta^2 \partial_\beta \bar{F}_\beta \,. \tag{2.7}$$

The partition function Z_β can be expressed in terms of the determinant of the field operator L_4 and the determinant can be usefully defined by using ζ-function [26]. In this way we get

$$\ln Z_\beta = \frac{1}{2}\zeta'_\beta(0|L_4/\mu^2) \,, \qquad \ln \bar{Z}_\beta = \frac{1}{2}\zeta'_\beta(0|\bar{L}_4/\mu^2) \,, \tag{2.8}$$

where μ is an arbitrary parameter necessary to adjust the dimensions and ζ' represents the derivative with respect to s of the ζ-function related to the operator in the argument.

3 Heat kernel and ζ-function in Rindler-like spacetimes

Here we resume the main results concerning the definition of free energy in Rindler-like spaces. A detailed analysis in arbitrary dimensions has been done in Ref. [21], where we refer the reader for more details.

We call Rindler-like spacetime a manifold of the form $\mathcal{M} = \mathcal{R} \times \mathcal{M}_2$, with the metric $ds^2 = ds^2(\mathcal{R}) + ds^2(\mathcal{M}_2)$, $\mathcal{R}$ being the 2-dimensional Rindler spacetime and $\mathcal{M}_2$ an arbitrary 2-dimensional smooth manifold. The Euclidean metric reads

$$ds^2 = x^2 d\tau^2 + dx^2 + \gamma_{ab}(y) dy^a dy^b \,,$$

where τ ($0 \le \tau \le \beta$) is the imaginary time, $x \ge 0$ the radial coordinate and y the coordinates on $\mathcal{M}_2$. For an arbitrary β the manifold $\mathcal{M}$ has the topology of $C_\beta \times \mathcal{M}_2$, C_β being the 2-dimensional cone. The Laplacian like operator assumes the form

$$L_4 = -\Delta_\beta + L_2 = -\Delta_\beta - \Delta_2 + f(y) \,, \tag{3.1}$$

where Δ_β and Δ_2 are the Laplace operators on C_β and $\mathcal{M}_2$ respectively and, for more generality, an arbitrary function on $\mathcal{M}_2$ has been added too.

The partition function on such kind of spaces can be written in the form [21]

$$F_\beta = -\frac{A_0(L_2) I_\beta(-1)}{4\beta\varepsilon^2} - \frac{A_1(L_2) I_\beta(0)}{2\beta} \ln\frac{\Lambda^2}{\varepsilon^2} + \frac{2\pi}{\beta} F_{2\pi} \,, \tag{3.2}$$

where ε and Λ are cutoff parameters and $A_n(L_2)$ are the spectral coefficients related to the Laplace-like operator L_2 on $\mathcal{M}_2$, that is

$$\operatorname{Tr} e^{-sL_2} \sim \sum_n A_n(L_2)\, s^{n-1} \,. \tag{3.3}$$

The function $F_{2\pi}$ is the free energy on the smooth manifold, that is in the absence of the conical singularity. It may have infrared (volume) divergences related to the cutoff Λ, but is regular for $\varepsilon \to 0$ (the horizon). Finally, the function $I_\beta(s)$ is strictly related to the ζ-function of the Laplacian on the cone and was studied in detail in Ref. [21] and in Ref. [27, that was called G_β]. Here we only need its values at $s = 0, -1$. They read

$$
\begin{aligned}
I_\beta(0) &= \frac{1}{6}\left(\frac{\beta}{2\pi} - \frac{2\pi}{\beta}\right) , && (3.4)\\
I_\beta(-1) &= \frac{1}{90}\left[\left(\frac{2\pi}{\beta}\right)^3 + 10\frac{2\pi}{\beta} - 11\frac{\beta}{2\pi}\right] . && (3.5)
\end{aligned}
$$

As expected, for $\beta = 2\pi$, $I_{2\pi}(0) = I_{2\pi}(-1) = 0$. In fact in this case the whoole manifold is smooth and one does not have singularities for $x \to 0$.

The formula for the entropy now reads

$$
S_\beta = \frac{A_0(L_2)}{90\varepsilon^2}\left[\left(\frac{2\pi}{\beta}\right)^3 + 5\,\frac{2\pi}{\beta}\right] - \frac{\pi A_1(L_2)}{3}\left(\frac{2\pi}{\beta}\right)\ln\frac{\Lambda}{\varepsilon} - 2\pi F_{2\pi} , \qquad (3.6)
$$

4 The Kerr's black hole

In this section we consider the Kerr solution of Einstein field equations and we study an approximation of the metric, which is valid near the horizon. It is suitable for the analysis of quantum field fluctuations near the black hole. The Kerr solution describes the gravitational field outside a rotating body (with axial symmetry) of mass M and angular momentum $J = Ma$ and is believed to be the unique solution for the description of all (uncharged) rotating black holes formed by collapse. It is usually written in the stationary form

$$
ds^2 = -\frac{\rho^2\Delta}{\Sigma^2}\,dt^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2 d\vartheta^2 + \frac{\Sigma^2}{\rho^2}\left(d\varphi - \frac{2aMr}{\Sigma^2}\,dt\right)^2 \sin^2\vartheta , \qquad (4.1)
$$

$$
\begin{aligned}
\Delta &= r^2 - 2Mr + a^2 = (r - r_+)(r - r_-) ,\\
\Sigma^2 &= (r^2 + a^2)^2 - a^2\Delta\sin^2\vartheta ,\\
\rho^2 &= r^2 + a^2\cos^2\vartheta .
\end{aligned}
$$

The irremovable singularity of space-time is given by $\rho^2 = 0$ and has the structure of a ring, while $r_\pm = M \pm \sqrt{M^2 - a^2}$ are singularities of the metric only. The important parameters which characterise such a solution are given by (in natural units $G = c = \hbar = k_B = 1$)

$$
\begin{aligned}
r_H &= r_+ = M + \sqrt{M^2 - a^2} , && \text{(event horizon position)},\\
A_H &= 4\pi(r_H^2 + a^2) = 8\pi M r_H , && \text{(area of the horizon)},\\
\kappa &= \frac{r_+ - r_-}{4Mr_+} , && \text{(surface gravity)}. \qquad (4.2)
\end{aligned}
$$

Another important region is the "ergosphere" where $g_{00} > 0$. It is given by

$$
r_H \le R_- < r < R_+ , \qquad R_\pm = M \pm \sqrt{M^2 - a^2\cos^2\vartheta} . \qquad (4.3)
$$

In such a region, a test particle at a fixed (coordinate) distance r from the body must rotate (with respect to the inertial observer at infinity) with angular velocity $\Omega(r) = \frac{2aMr}{\Sigma^2}$. The value of $\Omega(r)$ on the event horizon is identified with the angular velocity of the horizon itself Ω_H. It reads

$$
\Omega_H = \Omega(r_H) = \frac{2aMr_H}{\Sigma^2(r_H)} = \frac{a}{2Mr_H} . \qquad (4.4)
$$

Since we are interested in the physics outside and very near the black hole, we set

$$x^2 = \frac{4(r - r_+)}{r_+ - r_-} \tag{4.5}$$

and develope the metric for small x by taking into account only the leading contributions. The result is

$$ds_H^2 = \rho_H^2 \left(-\kappa^2 x^2\, dt^2 + dx^2 + d\vartheta^2\right) + \frac{\Sigma_H^2}{\rho_H^2} \sin^2\vartheta\, d\tilde{\varphi}^2 \,, \tag{4.6}$$

$$\tilde{\varphi} = \varphi - \Omega_H\, t \,,$$

$$\Sigma_H^2 = (r_H^2 + a^2)^2 = 4M^2 r_H^2 \,, \qquad \rho_H^2 = r_H^2 + a^2 \cos^2\vartheta \,,$$

By the suffix H we indicate the quantities evaluated on the horizon. They do not depend on x, but may depend on ϑ. Now, by the conformal transformation

$$\bar{g}_{\mu\nu} = \rho_H^{-2} g_{\mu\nu} \,, \qquad \sigma = -\ln \rho_H \,, \tag{4.7}$$

the line element in Eq. (4.6) assumes the Rindler-like form

$$d\bar{s}_H^2 = -\kappa^2 x^2\, dt^2 + dx^2 + d\vartheta^2 + f^2(\vartheta) \sin^2\vartheta\, d\tilde{\varphi}^2 \,, \tag{4.8}$$

$$f^2(\vartheta) = \left(\frac{r_H^2 + a^2}{r_H^2 + a^2\cos^2\vartheta}\right)^2 = 1 + \frac{a^2}{2M^2}\sin^2\vartheta + O\left([a/M]^4\right) \,. \tag{4.9}$$

To deal with finite temperature field theory, the imaginary time $\tau = it$ is assumed to be periodic with period β. For an arbitrary β the manifold has a conical singularity, which disappears if $\kappa\tau$ has a period equal to 2π. To be more precise, the manifold is smooth if the point $(\tau, x, \vartheta, \tilde{\varphi})$ is identified with the point $(\tau + 2\pi/\kappa, x, \vartheta, \tilde{\varphi})$. This means that, $(\tau, x, \vartheta, \varphi) \equiv (\tau + 2\pi/\kappa, x, \vartheta, \varphi - 2\pi\Omega_H/\kappa)$, in agreement with Gibbons-Hawking prescription [4]. To have an Euclidean section, also the replacement $a \to ia$, that is $\Omega \to i\Omega$, has to be performed. Then, the request that the manifold is smooth fixes the temperature $T = 1/\beta$ to the Hawking value [4]

$$T_H = \frac{\kappa}{2\pi} \,. \tag{4.10}$$

Now we can use the results of previous sections in order to compute the entropy of a scalar field near the horizon of a Kerr black hole. In has to be stressed that the results will be also valid for the Kerr-Newman geometry, since the charge simply modifies the form of the horizons $r_\pm$.

5 Thermodynamic of scalar fields in the Kerr's geometry

For convenience here we put $\kappa = 1$. The constant parameter κ will be easily restored at the end of calculations. We consider a minimally coupled, massive scalar field near the horizon of a Kerr black hole. The starting point is the D'Alembert operator in the original Kerr metric. Then we perform the approximation of the metric, Eq. (4.6), the Wick rotation and the conformal transformation (4.7) in this way arriving at

$$\bar{L}_4 = -\Delta_\beta + L_2 \,, \qquad L_2 = -\Delta_2 + \frac{\bar{R}}{6} - \Omega(\vartheta) \,, \tag{5.1}$$

$$\Omega(\vartheta) = \rho_H^2 \left(\frac{R_H}{6} - m^2\right) \,, \tag{5.2}$$

according to Eq. (2.4) with $\xi = 0$. In the latter equation, R_H and $\bar{R}$ are the scalar curvatures in the metrics ds_H^2 and $d\bar{s}_H^2$ respectively, Δ_β is the Laplace operator on the cone C_β, while Δ_2 is a Laplace-like operator on the smooth manifold $\mathcal{M}_2$, with metric

$$ds^2(\mathcal{M}_2) = d\vartheta^2 + f^2(\vartheta)\sin^2(\vartheta)\, d\tilde{\varphi}^2 \tag{5.3}$$

and curvature $R_2(\vartheta) = \bar{R}$. By a straightforward calculation one obtains

$$R_H = \frac{2}{\rho_H^2} - \frac{a^2 r_H^2 \sin^2\vartheta}{\rho_H^6}\,. \tag{5.4}$$

Since we only nead the first two spectral coefficients for the operator L_2, the explicit form of $\bar{R}$ is not strictly necessary and for this reason we do not write down it. The important thing is that it depends only on the coordinates of $\mathcal{M}_2$.

As is well known, the first two spectral coefficients are given by (for a review on spectral geometry see Ref. [28])

$$A_0(L_2) = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} f(\vartheta)\sin\vartheta\, d\vartheta\, d\tilde{\varphi} = \frac{2M}{a}\ \arctan\tfrac{a}{r_H}\,, \tag{5.5}$$

$$\begin{aligned} A_1(L_2) &= \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} \Omega(\vartheta) f(\vartheta)\sin\vartheta\, d\vartheta\, d\tilde{\varphi} \\ &= -2m^2 M r_H - \frac{M(r_H^2+3a^2)}{12 r_H(r_H^2+a^2)} - \frac{M(a^2-3r_H^2)}{4ar_H^2}\ \arctan\tfrac{a}{r_H}\,. \end{aligned} \tag{5.6}$$

Using the latter results in Eq. (3.6) at the equilibrium temperature $\beta = 2\pi$, we finally have

$$\begin{aligned} S_{T_H} \sim\ & \frac{A_H}{60\pi\varepsilon^2}\,\frac{1}{\kappa^2\, ar_H}\ \arctan\tfrac{a}{r_H} \\ & +\frac{1}{3}\left[\frac{m^2 A_H}{4\pi} + \frac{M(r_H^2+3a^2)}{12 r_H(r_H^2+a^2)} + \frac{M(a^2-3r_H^2)}{4ar_H^2}\ \arctan\tfrac{a}{r_H}\right]\ln\frac{\Lambda}{\varepsilon} \end{aligned} \tag{5.7}$$

where the constant κ has been reestablished. Of course, in the non rotating case $a \to 0$ we obtain the well known Schwarzschild result [6, 9]. The leading term of our expression is also compatible with the one given in Ref. [12].

We expect the latter equation for the entropy to be valid also for the Kerr-Newman black hole, since the charge simply modify the form of the horizons. This means that in the Kerr-Newman geometry the quantities r_H, A_H and κ depends also on the charge, but the form of the entropy is the same. This result is in contrast with an analog result given in Ref. [12]. In fact in that paper, the contribution to the logarithmic divergence due to the Kerr-Newman geometry is proportional to the charge and so, for Schwarzschild and Kerr black holes the logarithmic divergences are exactly the same. This is a very surprising result (according to the authors themselves) and in our opinion really strange since, as we said above, the charge enters only in expressions of $r_\pm$. Then we expect the final formulae to depend on the charge only throw $r_\pm$.

6 Conclusion

We have derived the one-loop quantum corrections to the entropy of a Kerr-Newman black hole due to a massive scalar field, using an approximation for the metric valid in a neighbourhood of the horizon. As expected, also at the Hawking temperature in the expression for the entropy there is a leading divergence, which goes as the inverse of the square of the distance from the horizon and a logarithmic divergence too. The leading term of Eq. (5.7) is compatible with the

analog expression obtained in Ref. [12], where a similar approximation for the metric has been used. As regards the logarithmic contribution, the two expressions are compatible only in the Schwarzschild case, while for the Kerr-Newman geometry they are completely different.

As a last comment we observe that the expression for the entropy obtained in Ref. [7] is more complicated than ours, since it contains also a cut-off, which regulates the integration in the ϑ angle and for this reason is really difficult to compare that expression with our.

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